Dealing with Incomplete Information in Linguistic Group Decision Making by Means of Interval Type 2 Fuzzy Sets

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Abstract

Nowadays in the social network based decision making processes, as the ones involved in e-commerce and e-democracy, multiple users with different backgrounds may take part and diverse alternatives might be involved. This diversity enriches the process but at the same time increases the uncertainty in the opinions. This uncertainty can be considered from two different perspectives: (i)the uncertainty in the meaning of the words given as preferences, that is motivated by the heterogeneity of the decision makers, (ii) the uncertainty inherent to any decision making process that may lead to an expert not being able to provide all their judgments. The main objective of this contribution is to address these two type of uncertainty. To do so the following approaches are proposed: Firstly, in order to capture, process and keep the uncertainty in the meaning of the linguistic assumption the Interval Type 2 Fuzzy Sets are introduced as a way to model the experts linguistic judgments. Secondly, a measure of the coherence of the information provided by each decision maker is proposed. Finally, a consistency based completion approach is introduced to deal with the uncertainty presented in the expert judgments. The proposed approach is tested in an e-democracy decision making scenario.

Keywords: Group decision making, Uncertainty, Linguistic preference relations , Incomplete information, Interval type 2 fuzzy sets, consistency , e-democracy.

1 Introduction

Decision making is one of the most pervasive task in people's daily routine. Usually to make a decision it is necessary to compare and judge the different alternatives to asses which is the best one. In most of these occasions people have to make perception-based rational decisions in environments of imprecision, uncertainty and partial truth. As a consequence, human judgments are frequently vague and subjective, making challenging to articulate opinions in a quantitative way. Obviously, a more natural alternative consists in using linguistic terms to describe the desired values. With this respect, Zadeh proposed the paradigm of *Computing With Words*, CWW,[48, 49] which models words by means of type-1 Fuzzy Sets (T1FS).

Nevertheless, various authors have expressed that T1FS presents some limitations[32] when modeling words. For instance, Herrera et al. remark in[20] the difficulty to find a membership function associated to a linguistic tag accepted by all the individuals. Going further Mendel remarks that "*words mean different things to different people and so are uncertain*",[30, 31] proving that modeling a word A using T1FS is not scientifically correct since the word would be well-defined by its membership function (MF) $\mu A(x)(x \in X)$ which is completely certain.[31] In this sense a useful tool could be modeling the words by means of Type 2 Fuzzy sets, that can contain infinite T1FS membership functions with infinite shapes and whose boundaries are defined buy two T1FS, the Upper membership function, UMF, and the lower membership function, LMF.

However, in spite of being proved more suitable for dealing with uncertainty, T2FS has not been extensively used in decision making mainly because of its computational complexity. A simplification that reduces dramatically the computational complexity and at the same time allows an optimal treatment of uncertainty consists on considering an uniform distribution of the uncertainty, leading to the concept on Interval Type 2 Fuzzy sets, IT2FS. IT2FS have attracted extensive researcher's attention in the field of decision making,[18, 25] having some of the classical decision making approaches such as TOPSIS[27, 38] and ELECTRE[10] adapted to the case when the preferences are expressed using IT2FS, or an interactive approach applied to medical decision making.[9] With this regard, a special mention requires Mendel and Wu Perceptual Computing Paradigm whose architecture, the Perceptual Computer, Per-C, is composed of three components:

- The encoder, in charge of transforming the words into IT2FSs providing the codebook-words with their associated FS models.
- Computer with words engine, CWW, that carrys out the computation of the different FS provided by the encoder and provides a resultant FS.
- The decoder, which maps the resultant FS from the CWW engine in a recommendation.

The Per-C has been adapted to carry out decision making process as an investment advisor and also in multicriteria and multiperson decision making for location choices[19] to mention some of its various applications.

A crucial issue in decision making is the consistency, that is, the coherence of the information determined by the absence of contradiction in each expert's given judgments.[1, 15] Obviously the less contradictory the information is, the more reliable and meaningful for the decision process. Consistency has been widely analyzed and modeled decision making approaches when the experts preferences are modeled by different types of linguistic and numeric preference relations. However no methodology has been proposed so far to asses the contradiction when the experts linguistic opinions are modeled by means of IT2FS.

Another key point in decision making scenarios is the incomplete information.[7, 37, 43] For example, in situations involving an elevated number of experts with different backgrounds and or a high number of alternatives,[4, 16, 41, 42, 50] there might be experts who present some hesitation in their judgments and therefore they are unable to pose an opinion for each one of the demanded comparison. To deal with these cases, there are some research approaches that propose to delete or penalize the information coming from those experts who provide incomplete preferences.[17, 35] However, as it has been demonstrated in[8] , incomplete preference relations, PR, derived from randomly deleting as much as 50 % of the elements of a complete pairwise PR provides good results without compromising accuracy and so discarding them could lead to the deletion of useful information generating biased and inaccurate results.[24] In order to estimate the incomplete information in decision making processes involving PRs a number of researches have been carried for various types of PRs, such as, Fuzzy PRs, Interval Value PRs, Intuitionistic PRs and Linguistic PRs modelled using T1FS and other ordinal models. An updated survey of these approaches have been reported in.[41] Nevertheless, to the extend of the authors knowledge, no methodology have been proposed to deal with incomplete preference relations when the linguistic information is modeled by means of IT2FS.

The main objectives of this contribution are twofold: First of all we define the consistency for the case of linguistic preference relations modeled by mean of IT2FS. This concept is based on the multiplicative transitivity. Build upon this we introduce a methodology to estimate the incomplete information using only the expert's preference relation.

This contribution is organized is the following way: In section 2 the main mathematical frameworks for modeling linguistic preferences relations by means of IT2FS in decision making among with basics concepts needed throughout the rest of the paper are discussed. In section 3 we address the issue of the consistency for linguistic preference relations modeled as IT2FS proposing a new cosistency measure.

Guided by this new consistency measure, in Section 4, an iterative completion process to estimate the incomplete information is proposed. The practical application of this approach is discussed in section 5 by means of an illustrative example. Finally, section 6 draws the conclusion pointing out some future research lines that this contribution opens.

2 Background

In decision making processes it has been observed that the pair-wise comparison of alternatives is one of the most effective methods of expressing opinions since it allows the evaluation of only two alternatives at a time,.[6,14,28,35] This comparison may results in three different output: the preference of one alternative, the state of indifference between them or the inability to compare them. This three different states have been merged in one unique concept of fuzzy preference relation[2] defined as follows:

Definition 1 (Preference Relation (PR)[36])**.** *A preference relation R is a binary relation defined on the set X that is characterized by a function $\mu_p : X \times X \rightarrow D$, where D is the domain of representation of preference degrees provided by the decision maker.*

R represents a $n \times n$ matrix $R = (r_{ij})$, where $r_{ij} = \mu_p(x_i, x_j)$ is the intensity of preference of alternative x_i over x_j. These elements can be numeric or linguistic, in what follows we focus on the second type.

2.1 Linguistic preference relations in decision making

Linguistic judgments in decision making may be modeled as an odd set of linguistic tags, $\mathcal{L} = \{l_0, \ldots, l_s | s \geq 2 \ \wedge i < j : \ l_i \ < \ l_j\}$, ordered in such a way that the central label $l_{s/2}$ symbolize the indifference in the comparison being the rest of the tags or labels placed in a symmetric way given the notion of transitivity.

Definition 2 (Linguistic Preference Relation (LPR)). *A LPR P on a finite set of alternatives X is characterized by a linguistic membership function $\mu_P \colon X \times X \longrightarrow \mathcal{L}$, $\mu_P(x_i, x_j) = p_{ij} \in \mathcal{L}$.*

There exist two widely accepted approaches to deal with LPRs in decision making contexts: (i) the cardinal representation which models the linguitic labels by means of fuzzy sets and their associated membership functions using as a reference the Zadeh's *extension principle*;[46] and (ii) the ordinal representation that uses the ordered structure of the labels to operate with the different judgments.[20,45] In this contribution we focus in the fist case.

2.1.1 Linguistic preference relation based on cardinal representation

In this case, each linguistic assessment is represented by means of a fuzzy number with an associated membership function that allocates for each value in $[0, 1]$ a degree of performance which represents its degree of compliance with the label .[47] As aforementioned, T1FS has grades of membership that are crisp, whereas in the case of T2FS, it contains fuzzy grades of membership. This type of representation may be useful when there exists uncertainty in the membership function for a fuzzy set, as it is the case of modeling a word.[29]

Definition 3. *A T2FS $\tilde{A}$ is a bivariate function on the Cartesian product, i.e., $\mu : X \times [0, 1]$ into $[0, 1]$, where X is the universe for primary variable of $\tilde{A}$, x. 3-D membership function of $\tilde{A}$ is usually denoted $\mu_{\tilde{A}}(x, u)$, where $x \in X$ and $u \in [0, 1]$, i.e.,*

$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u)) | x \in X, u \in [0, 1]\}, \tag{1}$$

in which $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$. $\tilde{A}$ can also be expressed as

$$\tilde{A} = \int_{x \in X} \int_{u \in [0,1]} \mu_{\tilde{A}}(x, u)/(x, u), \tag{2}$$

where $\int\int$ denotes union over all admissible x and u.

Definition 4. *When all $\mu_{\tilde{A}}(x, u) = 1$, then $\tilde{A}$ is an IT2FS.*

The IT2FS $\tilde{A}$ can be expressed as a special case of the T2FS in (3), represented as follows:[33]

$$\tilde{A} = \int_{x \in X} \int_{u \in [0,1]} 1/(x,u). \tag{3}$$

For universes of discourse X and U, $A_e = \int_{x \in X} u/x$ $(u \in [0,1])$ is called an embedded type-1 FS.

Since representing a three-dimensional figure of a T2 Membership function it is more complex than sketching two-dimensional figures of a T1 membership function a widely adopted way of representing a T2FS is by means of its footprint of uncertainty (FOU) on the two-dimensional domain of the T2FS.

Definition 5. *Uncertainty about $\tilde{A}$ is conveyed by the union of all its primary memberships μ_x, which is called the footprint of uncertainty (FOU) of $\tilde{A}$ (see Fig 1) that is,*

$$FOU(\tilde{A}) = \cup_{x \in X} \mu_x \tag{4}$$

Definition 6. *The upper membership function (UMF) and lower membership function (LMF) of $\tilde{A}$ are two type-1 MFs that bound the FOU. $UMF(\tilde{A})$ is associated with the upper bound of $FOU(\tilde{A})$ and*

$$UMF(\tilde{A}) \equiv \bar{\mu}_{\tilde{A}}(x) = (F\bar{O}U(\tilde{A}))\ \forall x \ \in \ X \tag{5}$$

$$LMF(\tilde{A}) \equiv \underline{\mu}_{\tilde{A}}(x) = (\underline{FOU}(\tilde{A}))\ \forall x \ \in \ X \tag{6}$$

In order to encode words into IT2FNs using IT2FS Liu and Mendel proposed the Interval Approach, IA in[26] where the interval endpoint data about a word are collected from a group of subjects by means of a survey. Then, each interval is mapped into a type-1 FS, and an IT2FS mathematical model (also represented by a FOU) is obtained for the word.

Definition 7. *If IT2FS $\tilde{a}$ can be expressed as:*

$$\tilde{a} = (a^U, a^L) = \left([a_1^U, a_2^U, a_3^U, a_4^U; h_a^U], [a_1^L, a_2^L, a_3^L, a_4^L; h_a^L]\right).$$

then $\tilde{a}$ is called the IT2TFN, where h_a^U denotes the membership value of the element a_i^U $(i = 2,3)$ in the upper trapezoidal membership function, and h_a^L denotes the membership value of the element a_i^L $(i = 2,3)$ in

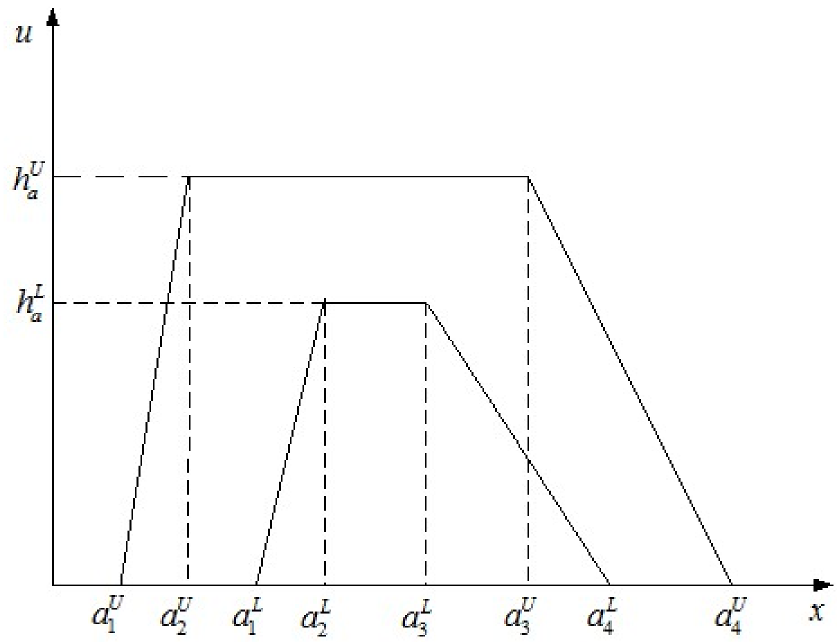

Figure 1: FOU for an IT2FN

the lower trapezoidal membership function, $h_a^U, h_a^L \in [0, 1]$*,* $i = 1, 2, 3, 4$.

Figure 1 depicts the FOU of an IT2FN $\tilde{a}$ with its upper and lower membership functions modelled as T1FS .

2.2 Extension Principle

The extension principle allows the functional translation from elements that are crisp to elements represented as fuzzy sets, as is demonstrated as follows:[34]

Definition 8 (Extension Principle)**.** *Let* $X_1 \times X_2 \times \ldots \times X_n$ *be a universal product set and* F *a functional mapping of the form*

$$F \colon X_1 \times X_2 \times \ldots \times X_n \longrightarrow Y$$

that maps the element $(x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \ldots \times X_n$ *to the element* $y = F(x_1, x_2, \ldots, x_n)$ *of the universal set* Y*. Let* A_i *be a fuzzy set over the universal set* X_i *with membership function* μ_{A_i} $(i = 1, 2, \ldots, n)$*. The membership function* μ_B *of the fuzzy set* $B = F(A_1, ., A_n)$ *over the universal set* Y *is:*

- *If* $\exists\ x_1, \ldots, x_n$ *such that* $y = F(x_1, \ldots, x_n)$:

$$\mu_B(y) = \sup_{y=F(x_1,x_2,\ldots,x_n)} \Big[\mu_{A_1}(x_1) * \mu_{A_2}(x_2) * \\ \ldots * \mu_{A_n}(x_n)\Big]$$

- *Otherwise:* $\mu_B(y) = 0$, *where* $*$ *is a t-norm.*

The expression in (13) involves the comparison of two products of three crisp numbers in the interval $[0, 1]$. The objective is to extend the function $f\colon [0,1] \times [0,1] \times\ [0,1] \longrightarrow [0,1]$,

$$f(x_1, x_2, x_3) = x_1 \cdot x_2 \cdot x_3, \tag{7}$$

to $f(C_1, C_2, C_3)$ being C_1, C_2, C_3 fuzzy sets over the set $[0, 1]$ with an associated membership functions μ_{C_1}, μ_{C_2}, μ_{C_3}, respectively.

According to the *extension principle*

$$H = f(C_1, C_2, C_3) \tag{8}$$

is a fuzzy set over the set $[0, 1]$ with membership function $\mu_H\colon [0,1] \to [0,1]$;

$$\mu_H(y) = \sup_{\substack{x_1 \cdot x_2 \cdot x_3 = y \\ x_1, x_2, x_3 \in [0,1]}} \left[\mu_{C_1}(x_1) \wedge \mu_{C_2}(x_2) \wedge \mu_{C_3}(x_3)\right].$$

where $(\wedge)$ represents the minimum t-norm operator.

According to the representation theorem fuzzy set can completely defined by decomposing it in its corresponding α-level sets.

An α-level set of a fuzzy set C over the universe L is defined as

$$C^\alpha = \{l \in L | \mu_C(l) \geq \alpha\}. \tag{9}$$

The set of crisp sets $\{C^\alpha | 0 < \alpha \leq 1\}$ is said to be a representation of the fuzzy set C. Indeed, the fuzzy set C can be represented as

$$C = \bigcup_{0<\alpha\leq 1} \alpha C^\alpha \tag{10}$$

with membership function

$$\mu_C(l) = \sup_{\alpha:\, l \in C_\alpha} \alpha. \tag{11}$$

Let C_1^α, C_2^α and C_3^α be the α-level sets of fuzzy sets C_1, C_2 and C_3 described above. We have

$$f\left(C_1^\alpha, C_2^\alpha, C_3^\alpha\right) = \Big\{x_1 \cdot x_2 \cdot x_3 | \\ x_1 \in C_1^\alpha, x_2 \in C_2^\alpha, x_3 \in C_3^\alpha\Big\}. \tag{12}$$

3 Consistency of fuzzy linguistic preference modeled as IT2FN relations

"Some individual opinions can be considered more consistent than other individual opinion",[13] where the explicit consistency can be defined as the "absence of explicit contradictions". Consistency is associated with the lack of contradiction in the information, and so with the quality of this information.[39] This concept has been extensively studied in decision making under fuzzy preference relations[22] concluding that the properties that ensure the existence of transitivity between the experts preferences may lead to consistency properties.[11] That is, if alternative a_i is preferred to a_j ($a_i \succ a_j$) and this one to a_k ($a_j \succ a_k$) then a_i should be preferred to a_k ($a_i \succ a_k$). This is known as weak stochastic transitivity. An extension of this transitivity has been presented by Tanino[40] in what it is called multiplicative transitivity.

Definition 9. *[Multiplicative Transitivity for Reciprocal fuzzy preference relation] A reciprocal fuzzy PR $P = (p_{ij})$ on a finite set of alternatives X is multiplicative transitive if and only if*

$$\forall i, k, j \in \{1, 2, \ldots n\}: \quad p_{ij} \cdot p_{jk} \cdot p_{ki} = p_{ik} \cdot p_{kj} \cdot p_{ji}. \tag{13}$$

When $p_{ij} > 0 \ \ \forall i, j$ it can be expressed as follows:

$$p_{ij} = \frac{p_{ik} \cdot p_{kj}}{p_{ik} \cdot p_{kj} + (1 - p_{ik}) \cdot (1 - p_{kj})}. \tag{14}$$

This property has been proved to extend weak stochastic transitivity, and so it implements the classical transitivity property for crisp values allowing the expression of preferences in the domain [0,1], instead of $\{0, 1\}$, which presents more restrictions.[11]

An extension of the multiplicative transitivity property for the case of linguistic PR represented by means af T1FS via both Zadeh's *Extension Principle* and the *Representation Theorem*[47] can be defined as follows:[12]

Definition 10 (Multiplicative transitivity of Linguistic PR as T1FS). *A fuzzy linguistic preference relation $R = (r_{ij})$ on a finite set of alternatives X is multiplicative transitive if and only if*

$$\forall \alpha \in (0,1] : f(r_{ij}^{\alpha}, r_{jk}^{\alpha}, r_{ki}^{\alpha}) = f(r_{ik}^{\alpha}, r_{kj}^{\alpha}, r_{ji}^{\alpha}) \quad \forall i,k,j. \tag{15}$$

In this case, the linguistic labels are characterized by T1FS in the unit interval, therefore, according to the representation theorem the α-level set of linguistic label r_{ij} is determined by closed interval: $r_{ij}^{\alpha} = [r_{ij}^{\alpha-}, r_{ij}^{\alpha+}]$.

Given that interval arithmetic follows the following:

$$f(r_{ij}^{\alpha}, r_{jk}^{\alpha}, r_{ki}^{\alpha}) = [r_{ij}^{\alpha-} \cdot r_{jk}^{\alpha-} \cdot r_{ki}^{\alpha-}, r_{ij}^{\alpha+} \cdot r_{jk}^{\alpha+} \cdot r_{ki}^{\alpha+}]. \tag{16}$$

then, we can express the previous definition as follows:

Definition 11 (Multiplicative transitivity of Linguistic PR as T1FS). *A fuzzy linguistic preference relation $R = (r_{ij})$ on a finite set of alternatives X is multiplicative transitive if and only if*

$$\forall \alpha \in (0,1] \wedge \forall i,k,j : \quad \left. \begin{aligned} r_{ij}^{\alpha-} \cdot r_{jk}^{\alpha-} \cdot r_{ki}^{\alpha-} &= r_{ik}^{\alpha-} \cdot r_{kj}^{\alpha-} \cdot r_{ji}^{\alpha-} \\ r_{ij}^{\alpha+} \cdot r_{jk}^{\alpha+} \cdot r_{ki}^{\alpha+} &= r_{ik}^{\alpha+} \cdot r_{kj}^{\alpha+} \cdot r_{ji}^{\alpha+} \end{aligned} \right\} \tag{17}$$

As aforementioned an IT2FS can be completely delimited between its UMF and its LMF having each of their membership functions modeled by mean of a T1FS, therefore the definition of multiplicative consistency for Linguistic preference relation modeled by mean of IT2FS can be extended from (17) as follows:

Definition 12 (Multiplicative transitivity of Linguistic PR as as IT2FS). *A fuzzy linguistic preference relation $R = (r_{ij}) = (r_{ij}^{L}, r_{ij}^{U})$ on a finite set of alternatives X is multiplicative transitive if and only if*

$\forall \alpha \in (0,1] \wedge \forall i,k,j$:

$$\left.\begin{array}{l} r_{ij}^{L\alpha-} \cdot r_{jk}^{L\alpha-} \cdot r_{ki}^{L\alpha-} = r_{ik}^{L\alpha-} \cdot r_{kj}^{L\alpha-} \cdot r_{ji}^{L\alpha-} \\ r_{ij}^{L\alpha+} \cdot r_{jk}^{L\alpha+} \cdot r_{ki}^{L\alpha+} = r_{ik}^{L\alpha+} \cdot r_{kj}^{L\alpha+} \cdot r_{ji}^{L\alpha+} \\ r_{ij}^{U\alpha-} \cdot r_{jk}^{U\alpha-} \cdot r_{ki}^{U\alpha-} = r_{ik}^{U\alpha-} \cdot r_{kj}^{U\alpha-} \cdot r_{ji}^{U\alpha-} \\ r_{ij}^{U\alpha+} \cdot r_{jk}^{U\alpha+} \cdot r_{ki}^{U\alpha+} = r_{ik}^{U\alpha+} \cdot r_{kj}^{U\alpha+} \cdot r_{ji}^{U\alpha+} \end{array}\right\} \tag{18}$$

3.1 Consistency Level

We can take advantage of the aforementioned Tanino's Multiplicative consistency property (14) in order to estimate a fully consistent preference value between a pair of alternatives (a_i, a_j) using an intermediate alternative a_k $(k \neq i, j)$ in the following way:

$$est_{ij}^k = \frac{p_{ik} \cdot p_{kj} \cdot p_{ji}}{p_{jk} \cdot p_{ki}} \tag{19}$$

Notice that as long as the denominator is not zero est_{ij}^k can be considered as one of the multiplicative transitivity based estimated fuzzy preference value for alternatives (a_i, a_j) calculated by means of the intermediate alternative a_k. The average of all these partially multiplicative transitivity values can be interpreted as the global multiplicative transitivity estimated value as it expressed as follows:

$$est_{ij} = \frac{\sum_{k \in P_{ij}^{01}} est_{ij}^k}{\#P_{ij}^{01}};$$

where $P_{ij}^{01} = \{k \neq i, j | (p_{ik}, p_{kj}) \notin P^{01}\}$, $P^{01} = \{(1,0),(0,1)\}$, and $\#P_{ij}^{01}$ is the cardinality of P_{ij}^{01}.

Following this reasoning we can conclude that for every given fuzzy PR, $P = (p_{ij})$ a multiplicative transitivity fuzzy PR, $MT = (mest_{ij})$ can be derived and so we can consider that $P = (p_{ij})$ is multiplicative transitive when $P = MT$ since if P is multiplicative transitive then (14) holds $\forall i, j, k$.

Definition 13. *[Multiplicative Consistency for fuzzy PR] A fuzzy PR $R = (r_{ij})$ is multiplicative consistent if and only if $R = MT$.*

The degree of similarity existing between the expert's matrix of preference P and MT is used as a measure of the level of consistency of a fuzzy PR.[22]

3.2 Consistency Level for Linguistic preference relations

In the following the concept of consistency by means of linguistic transitivity is extended to the case of Linguistic PRs expressed as T1FS and IT2FS.

For the case of Linguistic labels characterized by T1FS and IT2FS in the unit interval we also take advantage of the multiplicative consistency defined in (17) and (18) respectively to obtain, by means of an intermediate preference relation, the partial multiplicative transitivity estimated value mr_{ij}^{k}.

Consistency Index of Linguistic preferences modeled as T1FS

In the case of LPRs expressed as T1FS we compute the corresponding α-level set of the partial multiplicative transitivity estimated value as follows:

$$\forall \alpha \in (0,1] \wedge \forall i,j,k: \quad \left.\begin{array}{l} mr_{ij}^{k\alpha-} = \begin{cases} 0, & (r_{ik}^{\alpha-}, r_{ji}^{\alpha-}) \in \{(0,1),(1,0)\} \\ \dfrac{r_{ik}^{\alpha-} \cdot r_{ji}^{\alpha-}}{r_{ik}^{\alpha-} \cdot r_{ji}^{\alpha-} + (1 - r_{ik}^{\alpha-}) \cdot (1 - r_{ji}^{\alpha-})}, & \text{Otherwise.} \end{cases} \\ mr_{ij}^{k\alpha+} = \begin{cases} 0, & (r_{ik}^{\alpha+}, r_{ji}^{\alpha+}) \in \{(0,1),(1,0)\} \\ \dfrac{r_{ik}^{\alpha+} \cdot r_{ji}^{\alpha+}}{r_{ik}^{\alpha+} \cdot r_{ji}^{\alpha+} + (1 - r_{ik}^{\alpha+}) \cdot (1 - r_{ji}^{\alpha+})}, & \text{Otherwise.} \end{cases} \end{array}\right\} \tag{20}$$

The fully consistent estimated value mr_{ij}^{α} of r_{ij}^{α} is obtained as the average of all possible $mr_{ij}^{k\alpha}$:

$$\forall \alpha \in (0,1] \wedge \forall i,j,k: \quad \left.\begin{array}{l} mr_{ij}^{\alpha-} = \frac{\sum_{j=1; i \neq k \neq j}^{n} mr_{ij}^{k\alpha-}}{n-2} \\ mr_{ij}^{\alpha+} = \frac{\sum_{j=1; i \neq k \neq j}^{n} cp_{ij}^{k\alpha+}}{n-2} \end{array}\right\} \tag{21}$$

$$CL_{ij} = sim(r_{ij}, mr_{ij}) \quad \forall i,j \tag{22}$$

Here $sim(r_{ij}, cp_{ij})$ represents the similarity measure between the values r_{ij} and cp_{ij}. In this case both preference relations are T1FS instead of crips values. There exists several similarity measures for T1FSs. In this contribution we use the extension of the Jacquard similarity measure that satisfies reflexivity, symmetry, transitivity and overlapping.

Definition 14. ***Jaccard simmilarity measure.***[23]

$$sim_j(A, B) = \frac{f(A \cap B)}{f(A \cup B)} \tag{23}$$

Definition 15. ***Jaccard simmilarity measure for T1FS***[29]***.***

Given a fuzzy linguistic preference relation $R = (r_{ij})$ and a its corresponding fully consistent linguistic preference relation $CP = (cp_{ij})$. The Jaccard similarity between both of them is:

$$sim(r_{ij}, cp_{ij}) = \frac{\sum_{k=1}^{N} min(r_{ij}(x_k), cp_{ij}(x_k))}{\sum_{k=1}^{N} max(r_{ij}(x_k), cp_{ij}(x_k))} \tag{24}$$

where $x_k(k = 1, \cdots, N)$ are equally spaced in the support $r_{ij} \cup cp_{ij}$

Consistency Index for Linguistic preferences modeled as IT2FS

In the case of Linguistic Preferences modeled as IT2FS, as aforementioned, an IT2FS is composed of two T1FS, the UMF function and the LMF, $r_{ik} = (r_{ik}^L, r_{ik}^U)$. Whose membership functions can be expressed by mean of its corresponding α-level set, in which each member of the set is represented as $r_{ij}^{\alpha} = ([r_{ij}^{L\alpha-}, r_{ij}^{L\alpha+}], [r_{ij}^{U\alpha-}, r_{ij}^{U\alpha+}])$

Therefore to obtain the partial multiplicative transitivity based value $mr_{ij}^k = (mr_{ij}^{kL}, r_{ij}^{kU})$ we can compute its corresponding α-level set for each one of the UMF and the LMF as follows:

$\forall \alpha \in (0,1] \wedge \forall i,k,j:$

$$\left.\begin{array}{l}
mr_{ij}^{kL\alpha-} = \begin{cases} 0, & (r_{ik}^{L\alpha-}, r_{ji}^{L\alpha-}) \in \{(0,1),(1,0)\} \\ \dfrac{r_{ik}^{L\alpha-} \cdot r_{ji}^{L\alpha-}}{r_{ik}^{L\alpha-} \cdot r_{ji}^{L\alpha-} + (1 - r_{ik}^{L\alpha-}) \cdot (1 - r_{ji}^{L\alpha-})}, & \text{Otherwise.} \end{cases} \\
mr_{ij}^{kL\alpha+} = \begin{cases} 0, & (r_{ik}^{L\alpha+}, r_{ji}^{L\alpha+}) \in \{(0,1),(1,0)\} \\ \dfrac{r_{ik}^{L\alpha+} \cdot r_{ji}^{L\alpha+}}{r_{ik}^{L\alpha+} \cdot r_{ji}^{L\alpha+} + (1 - r_{ik}^{L\alpha+}) \cdot (1 - r_{ji}^{L\alpha+})}, & \text{Otherwise.} \end{cases} \\
mr_{ij}^{kL\alpha-} = \begin{cases} 0, & (r_{ik}^{U\alpha-}, r_{ji}^{U\alpha-}) \in \{(0,1),(1,0)\} \\ \dfrac{r_{ik}^{U\alpha-} \cdot r_{ji}^{U\alpha-}}{r_{ik}^{U\alpha-} \cdot r_{ji}^{U\alpha-} + (1 - r_{ik}^{U\alpha-}) \cdot (1 - r_{ji}^{U\alpha-})}, & \text{Otherwise.} \end{cases} \\
mr_{ij}^{kL\alpha+} = \begin{cases} 0, & (r_{ik}^{U\alpha+}, r_{ji}^{U\alpha+}) \in \{(0,1),(1,0)\} \\ \dfrac{r_{ik}^{U\alpha+} \cdot r_{ji}^{U\alpha+}}{r_{ik}^{U\alpha+} \cdot r_{ji}^{U\alpha+} + (1 - r_{ik}^{U\alpha+}) \cdot (1 - r_{ji}^{U\alpha+})}, & \text{Otherwise.} \end{cases}
\end{array}\right\} \tag{25}$$

The overall multiplicative transitivity based estimated value mr_{ij}^{α} of r_{ij}^{α} is obtained as the average of all possible $mr_{ij}^{k\alpha}$:

$$\forall \alpha \in (0,1] \wedge \forall i,k,j:$$

$$\left.\begin{array}{l}
mr_{ij}^{L\alpha-} = \frac{\sum_{k=1;i\neq k\neq j}^{n} mr_{ij}^{Lk\alpha-}}{n-2} \\
mr_{ij}^{L\alpha+} = \frac{\sum_{k=1;i\neq k\neq j}^{n} mr_{ij}^{Lk\alpha+}}{n-2} \\
mr_{ij}^{U\alpha-} = \frac{\sum_{k=1;i\neq k\neq j}^{n} mr_{ij}^{Uk\alpha-}}{n-2} \\
mr_{ij}^{U\alpha+} = \frac{\sum_{k=1;i\neq k\neq j}^{n} mr_{ij}^{Uk\alpha+}}{n-2}
\end{array}\right\} \tag{26}$$

To compute consistency index at level 1 for the case of IT2FS we first have to obtain the similarity measure in (22). In the literature one can find seven similarity measures for IT2FS, each one presenting different drawbacks.[29] Among them, we choose the extension of the Jaccard's distance for IT2FS proposed in[29] to be the only one that satisfies at the same time the desired properties of transitivity, reflexivity symmetry and overlapping.

Definition 16. ***Jaccard similarity measure for IT2FS[29]***

Given a fuzzy linguistic preference relation $R = (r_{ij}) = (r_{ij}^L, r_{ij}^U)$ and a its corresponding fully consistent *linguistic preference relation* $CP = (cp_{ij}) = (cp^L_{ij}, cp^U_{ij})$. *The Jaccard similarity between both of them is:*

$$sim(r_{ij}, cp_{ij}) = \frac{\sum_{k=1}^{N} min(r^U_{ij}(x_k), cp^U_{ij}(x_k)) + \sum_{k=1}^{N} min(r^L_{ij}(x_k), cp^L_{ij}(x_k))}{\sum_{k=1}^{N} max(r^U_{ij}(x_k), cp^U_{ij}(x_k)) + \sum_{k=1}^{N} max(r^L_{ij}(x_k), cp^L_{ij}(x_k))} \quad (27)$$

where $x_k (k = 1, \cdots, N)$ *are equally spaced in the support* $r_{ij} \cup cp_{ij}$

Once we have computed the consistency Level 1, that is, the Consistency Index of pair of alternatives, for Linguistic preferences modeled as T1FS, and IT2FS the subsequent consistency levels can be directly calculated by means of the Jaccard distance in (27).

4 Proposed completion method

In order to model the situations in which an expert is not able to provide all the judgements about all the pairwise comparisons of the alternatives, the concept of incomplete preference relation has been proposed in[22] and it is defined as follows:

Definition 17. *A function* $f: X \longrightarrow Y$ *is* partial *when not every element in the set* X *necessarily maps to an element in the set* Y. *When every element from the set* X *maps to one element of the set* Y *then we have a* total *function.*

Definition 18. *A preference relation* P *on a set of alternatives* X *with a* partial *membership function is an* incomplete preference relation.

In this contribution we propose an approach that estimates the missing linguistic judgments when they are modeled by means of T1FS and IT2Fs. The proposed approach consists on a sequential iterative procedure that for estimating each unknown linguistic judgment r_{ik} $(i \neq j)$ uses the known intermediate preference values (r_{ij}, r_{jk}), to derive the local estimated values driven by the multiplicative consistency property introduced in the previous section.

Firstly, we introduce the notation that we are going to follow in the procedure:

$$
\begin{aligned}
&A = \{(i,j) \mid i,j \in \{1,\dots,n\} \wedge i \neq j\} \\
&X = \{(i,j) \in A \mid p_{ij} \text{ is unknown}\} \\
&KW^h = A \setminus MV^h \\
&C_{ik}^1 = \{j \neq i,k \mid (i,j),(j,k) \in KW\} \\
&C_{ik}^2 = \{j \neq i,k \mid (j,i),(j,k) \in KW\} \\
&C_{ik}^3 = \{j \neq i,k \mid (i,j),(k,j) \in KW\} \\
&EX_i = \{(a,b) \mid (a,b) \in KW \wedge (a = i \vee b = i)\},
\end{aligned}
\tag{28}
$$

where KW^h consists on the the set of pairs of alternatives provided by the given expert and X is the set alternatives to that are not given and so they have to be estimated. Finally, C_{ik}^{h1}, C_{ik}^{h2}, C_{ik}^{h3} consists on the given alternatives x_j $(j \neq i,k)$ that are used to estimate the missing values p_{ik} $(i \neq k)$ as it is explained in the following subsections.

In each iteration t, the procedure selects the unknown preferences X, that have enough data to be estimated EX_t in the following way:

$$
EX_t = \left\{ (i,k) \in X \setminus \bigcup_{l=0}^{t-1} EX_l \mid i \neq k \wedge \exists j \in \{C_{ik}^1 \cup C_{ik}^2 \cup C_{ik}^3\} \right\}, \tag{29}
$$

The steps that comply the procedure to estimate a particular value p_{ik} with $(i,k) \in EX_t$ are the following:

Step 1 $(cp_{ik})^1 = (s_0, 0)$, $(cp_{ik})^2 = (s_0, 0)$, $(cp_{ik})^3 = (s_0, 0)$, $\mathcal{K} = 0$.

Step 2 if $\#C_{ik}^1 \neq 0$, then $(cp_{ik})^1 = \sum_{j \in C_{ik}^1} ((cp_{ik})^{j1})/\#C_{ik}^1$, $\mathcal{K}++$.

Step 3 if $\#C_{ik}^2 \neq 0$, then $(cp_{ik}^h)^2 = \sum_{j \in C_{ik}^2} ((cp_{ik})^{j2})/\#C_{ik}^2$, $\mathcal{K}++$.

Step 4 if $\#C_{ik}^3 \neq 0$, then $(cp_{ik})^3 = \sum_{j \in C_{ik}^3} ((cp_{ik})^{j3})/\#C_{ik}^3$, $\mathcal{K}++$.

Step 5 Calculate $cp_{ik} = \left(\frac{(cp_{ik})^1 + cp_{ik})^2 + (cp_{ik})^3}{\mathcal{K}} \right)$.

Once the iterative process to estimate one missing preference relation solely from the experts given values is clarified, it is necessary to determine how the value of cp_{ik} can be estimated: In the following we explain how to estimate the missing values at each iteration t of the process for Linguistic preference relations modeled as T1FS and as IT2FS.

4.1 Completion approach for LPRs modeled as T1FS

Given a missing preference relation r_{ik} we can compute its corresponding α-level set, $r_{ik}^{\alpha} = [r_{ik}^{\alpha-}, r_{ij}^{\alpha+}]$, as follows:

$$\forall \alpha \in (0,1] \wedge \forall i,k,j: \left.\begin{array}{l} cp_{ik}^{j\alpha-} = \begin{cases} 0, & (r_{ij}^{\alpha-}, r_{jk}^{\alpha-}) \in \{(0,1),(1,0)\} \\ \dfrac{r_{ij}^{\alpha-} \cdot r_{jk}^{\alpha-}}{r_{ij}^{\alpha-} \cdot r_{jk}^{\alpha-} + (1 - r_{ij}^{\alpha-}) \cdot (1 - r_{jk}^{\alpha-})}, & \text{Otherwise.} \end{cases} \\ cp_{ik}^{j\alpha+} = \begin{cases} 0, & (r_{ij}^{\alpha+}, r_{jk}^{\alpha+}) \in \{(0,1),(1,0)\} \\ \dfrac{r_{ij}^{\alpha+} \cdot r_{jk}^{\alpha+}}{r_{ij}^{\alpha+} \cdot r_{jk}^{\alpha+} + (1 - r_{ij}^{\alpha+}) \cdot (1 - r_{jk}^{\alpha+})}, & \text{Otherwise.} \end{cases} \end{array}\right\} \quad (30)$$

The overall estimated value for this concrete α-level cp_{ik}^{α} of r_{ik}^{α} is obtained as the average of all possible $cp_{ik}^{j\alpha}$, whose cardinality is $\mathcal{K}$:

$$\forall \alpha \in (0,1] \wedge \forall i,k,j: \left.\begin{array}{l} cp_{ik}^{\alpha-} = \frac{\sum_{j=1; i \neq k \neq j}^{n} cp_{ik}^{j\alpha-}}{\mathcal{K}} \\ cp_{ik}^{\alpha+} = \frac{\sum_{j=1; i \neq k \neq j}^{n} cp_{ik}^{j\alpha+}}{\mathcal{K}} \end{array}\right\} \quad (31)$$

4.2 Completion approach for LPR modelled as IT2FS

As aforementioned, the FOU of an IT2FS is composed of two T1FS, the UMF function and the LMF, $r_{ik} = (r_{ik}^{L}, r_{ik}^{U})$. Moreover each of the membership functions can be expressed by mean of its corresponding α-level set in which each member of the set is represented as follows: $r_{ik}^{\alpha} = ([r_{ik}^{L\alpha-}, r_{ik}^{L\alpha-}], [r_{ik}^{U\alpha-}, r_{ik}^{U\alpha+}])$

Therefore to estimate a missing preference relation $r_{ik} = (r_{ik}^{L}, r_{ik}^{U})$ we can compute its corresponding α-level set for each one of the UMF and the LMF as follows:

$$\forall \alpha \in (0,1] \wedge \forall i,k,j:$$

$$\left.\begin{array}{l}
cp_{ik}^{Lj\alpha-} = \begin{cases} 0, & (r_{ij}^{L\alpha-}, r_{jk}^{L\alpha-}) \in \{(0,1),(1,0)\} \\ \dfrac{r_{ij}^{L\alpha-} \cdot r_{jk}^{L\alpha-}}{r_{ij}^{L\alpha-} \cdot r_{jk}^{L\alpha-} + (1 - r_{ij}^{L\alpha-}) \cdot (1 - r_{jk}^{L\alpha-})}, & \text{Otherwise.} \end{cases} \\
cp_{ik}^{Lj\alpha+} = \begin{cases} 0, & (r_{ij}^{L\alpha+}, r_{jk}^{L\alpha+}) \in \{(0,1),(1,0)\} \\ \dfrac{r_{ij}^{L\alpha+} \cdot r_{jk}^{L\alpha+}}{r_{ij}^{L\alpha+} \cdot r_{jk}^{L\alpha+} + (1 - r_{ij}^{L\alpha+}) \cdot (1 - r_{jk}^{L\alpha+})}, & \text{Otherwise.} \end{cases} \\
cp_{ik}^{Uj\alpha-} = \begin{cases} 0, & (r_{ij}^{U\alpha-}, r_{jk}^{U\alpha-}) \in \{(0,1),(1,0)\} \\ \dfrac{r_{ij}^{U\alpha-} \cdot r_{jk}^{U\alpha-}}{r_{ij}^{U\alpha-} \cdot r_{jk}^{U\alpha-} + (1 - r_{ij}^{U\alpha-}) \cdot (1 - r_{jk}^{U\alpha-})}, & \text{Otherwise.} \end{cases} \\
cp_{ik}^{Uj\alpha+} = \begin{cases} 0, & (r_{ij}^{U\alpha+}, r_{jk}^{U\alpha+}) \in \{(0,1),(1,0)\} \\ \dfrac{r_{ij}^{U\alpha+} \cdot r_{jk}^{U\alpha+}}{r_{ij}^{U\alpha+} \cdot r_{jk}^{U\alpha+} + (1 - r_{ij}^{U\alpha+}) \cdot (1 - r_{jk}^{U\alpha+})}, & \text{Otherwise.} \end{cases}
\end{array}\right\} \tag{32}$$

The overall estimated value cp_{ik}^{α} of r_{ik}^{α} is obtained as the average of all possible $cp_{ik}^{j\alpha}$:

$$\forall \alpha \in (0,1] \wedge \forall i,k,j:$$

$$\left.\begin{array}{l}
cp_{ik}^{L\alpha-} = \frac{\sum_{j=1; i \neq k \neq j}^{n} cp_{ik}^{Lj\alpha-}}{\mathcal{K}} \\
cp_{ik}^{L\alpha+} = \frac{\sum_{j=1; i \neq k \neq j}^{n} cp_{ik}^{Lj\alpha+}}{\mathcal{K}} \\
cp_{ik}^{U\alpha-} = \frac{\sum_{j=1; i \neq k \neq j}^{n} cp_{ik}^{Uj\alpha-}}{\mathcal{K}} \\
cp_{ik}^{U\alpha+} = \frac{\sum_{j=1; i \neq k \neq j}^{n} cp_{ik}^{Uj\alpha+}}{\mathcal{K}}
\end{array}\right\} \tag{33}$$

This approach allows to estimate the missing preferences as long as there is one at least one preference value involving the alternative. Therefore, the sufficient condition for an incomplete LPR to be completed, is that the experts provide a set of $n-1$ non-leading diagonal PR, in which there is at least on comparative judgment for each of the alternatives.[22]

5 Example

Let us suppose the city council decides to carry out an online process to ask the citizens about which is the best neighborhood to place a new park. In this case, in order to facilitate the process the people taking part in the decision will express their opinions by means of the following linguistic term set.

Table 1: Linguistic Term set

Linguistic Terms	Abbreviations
Null	N
Much Worse	MW
Worse	W
Slightly Worse	SW
Indiference	IF
Slightly Better	SB
Better	B
Much Better	MB
Absolutely Preferred	P

First of all, the preference values for each pairwise comparison must be transformed into IT2TFNs. To do so, the Enhanced Interval Approach, EIA,,[44] that is composed of two main parts, the data part and the fuzzy set part, is applied.

In order to get the meaning of the words in terms of interval suitable as input for the EIA we have performed a random survey for this 8 linguistic terms listed in Table 5 in which the interval datasets were collected from 20 students.

Table 5 shows the resulting FOU for each linguistic term and their graphical representation are depicted in Figure 2.

Table 2: FOU data for all linguistic terms.

Words	UMF	LMF	Centroid	Center of centroid
N	[0, 0,0,0;0]	[0, 0,0,0;0]	[0,0]	0
MW	[0.10,0.10,0,17,0.26;1]	[0.10,0.10,0.11,0.19;1]	[0.13,0.16]	0.148
W	[0.14,0.24,0.28,0.34;1]	[0.25,0.27,0.27,0.31;0.63]	[0.22,0.29]	0.26
SW	[0.22,0.30,0.39,0.48;1]	[0.36,0.35,0.35,0.38;0.3673]	[0.28,0.41]	0.34
IF	[0.33,0.43,0.50,0.60;1]	[0.44,0.47,0.47,0.50;0.43]	[0.40,0.52]	0.467
SB	[0.47,0.55,0.62,0.69;1]	[0.56,0.59,5.89,0.61;0.45]	[0.53,0.64]	0.58
B	[0.59,0.67,0.71,0.82;1]	[0.67,0.69,0.69,0.71;0.61]	[0.65,0.75]	0.71
MB	[0.70,0.77,0.80,0.88;1]	[0.75,0.78,0.78,0.81;0.73]	[0.76,0.81]	0.79
P	[0.70,0.89,0.90,0.90;1]	[0.89,0.90,0.90,0.90;1.00]	[0.79,0.89]	0.84

Let $X = \{x_1, x_2, x_3, x_4\}$ be the set of alternative locations evaluated by one of the decision makers who presents the following incomplete linguistic preference relation R.

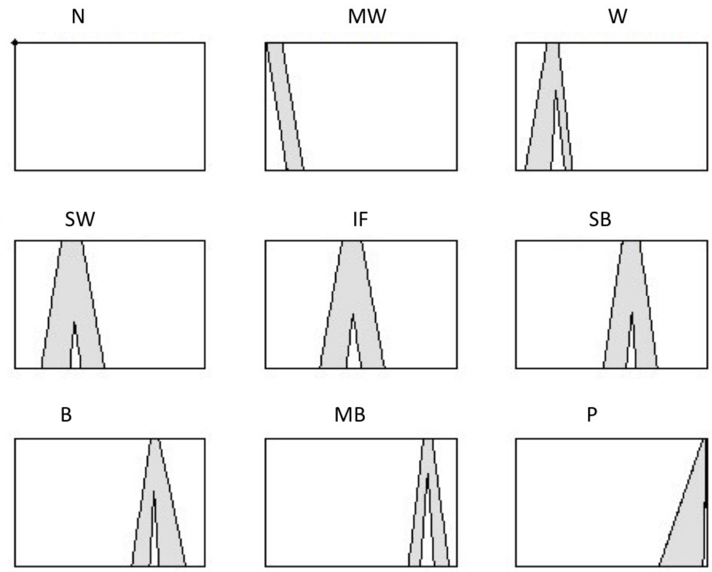

Figure 2: Linguistic Labels expressed by means of IT2FS

$$R = \begin{pmatrix} - & MW & x & x \\ MB & - & SW & x \\ x & SB & - & P \\ x & x & N & - \end{pmatrix}$$

The equivalent IT2FPR is represented as follows by means of its UMF and its LMF.

$$R^U = \begin{pmatrix} - & [0.10, 0.10, 0, 17, 0.26; 1] & x & x \\ 0.70, 0.77, 0.80, 0.88; 1] & - & [0.22, 0.30, 0.39, 0.48; 1] & x \\ x & [0.47, 0.55, 0.62, 0.69; 1] & - & [0.70, 0.89, 0.90, 0.90; 1] \\ x & x & [0, 0, 0, 0; 0] & - \end{pmatrix}$$

$$R^L = \begin{pmatrix} - & [0.10, 0.10, 0.11, 0.19; 1] & x & x \\ 0.75, 0.78, 0.78, 0.81; 0.73] & - & [0.36, 0.35, 0.35, 0.38; 0.3673] & x \\ x & [0.47, 0.55, 0.62, 0.69; 1] & - & [0.70, 0.89, 0.90, 0.90; 1] \\ x & x & [0, 0, 0, 0; 0] & - \end{pmatrix}$$

In the following we are going to estimate the missing values for this given linguistic matrix of preference. To do so, the iterative procedure detailed in the previous section is going to be applied.

Step 1: In this first step the set of PRs that can be computed as follows.

$$EMV_1 = \{(1,3),(2,4),(3,1),(4,2)\}.$$

In this case the computation of the element $(1,4)$ cannot be carried out since there are not any intermediate judgement.

The computation of the estimated UPM b_{13}^U and LMF b_{13}^L is given below. These values are computed using the available preferences k in which the PR $(1,k)$ and $(k,3)$ are known. In this case the only PR available is when $k=2$, resulting in the following (rounding to 2 decimal places):

$$b_{13}^U = b_{13}^{U2} = \frac{b_{12}^U \cdot b_{23}^U}{b_{12}^U \cdot b_{23}^U + (1-b_{12}^U)\cdot(1-b_{23}^U)} = [0.03, 0.05, 0.12, 0.24; 1]$$

and

$$b_{13}^L = b_{13}^{L2} = \frac{b_{12}^L \cdot b_{23}^L}{b_{12}^L \cdot b_{23}^L + (1-b_{12}^L)\cdot(1-b_{23}^L)} = [0.060.060.060.13; 1]$$

Step 2: At this step we can estimate the rest of the missing values. $EMV_2 = \{(1,4),(4,1)\}$. For the case of r_{14} the computation process is as follows:

$$b_{14}^U = b_{14}^{U3} = \frac{b_{13}^U \cdot b_{34}^U}{b_{13}^U \cdot b_{34}^U - (1-b_{13}^U)\cdot(1-b_{34}^U)} = [0.07, 0.28, 0.54, 0.74; 1]$$

and

$$b_{14}^L = b_{14}^{L3} = \frac{b_{13}^L \cdot b_{34}^L}{b_{13}^L \cdot b_{34}^L + (1-b_{13}^L)\cdot(1-b_{34}^L)} = [0.34, 0.35, 0.37, 0.56; 1]$$

The rest of values can be estimated following a similar computation process:

$$r_{24}^U = [0.15, 0.56, 0.82, 0.96; 1]$$

$$r_{24}^L = [0.6, 0.66, 0.68, 0.85; 1]$$

$$r_{31}^U = [0.67, 0.8, 0.87, 0.94; 1]$$

$$r_{31}^L = [0.79, 0.84, 0.84, 0.87; 0.69]$$

$$r_{41}^U = [0.19, 0.31, 0.57, 0.85; 1]$$

$$r_{41}^L = [0.3, 0.36, 0.39, 0.61; 1]$$

$$r_{42}^U = [0.09, 0.12, 0.25, 0.44; 1]$$

$$r_{42}^L = [0.12, 0.14, 0.15, 0.27; 1]$$

6 Conclusions

Prof. Zadeh coined the term *computing with words* that consists in a methodology in which the objects of computation are words and propositions drawn from a natural language. However, according to Prof. Mendel: *A word may mean different things to different people*, and so this uncertainty in the meaning should be addressed in the group decision processes in which several decision makers pose their opinions by means of words. In this contribution, we have presented a new group decision making methodology in which the linguistic labels provided by the experts are modeled by means of IT2FS in order to capture this uncertainty in the meanings of the words. To do so, firstly we have proposed a measure to asses the quality of the information provided by each expert developing the concept of consistency for the case of linguistic preferences expressed by means of IT2FS. Secondly, a new approach to estimate the missing linguistic information using a consistency based process is presented.

The main novelty of the proposed approach is that it deals with the uncertainty inherent in any decision making process in two different ways, the uncertainty in the significance of the words and the uncertainty in the experts opinions that may deal to incomplete information. This procedure simplifies to the experts their opinion representation, allowing linguistic judgments, but at the same time it is able to capture and process the uncertainty in the meaning during all the decision making process. Therefore, this approach is of utility in large scale decision making processes as the ones involving e-commerce and e-democracy in which a large number of heterogeneous users are asked to provide their judgments.

As future word we plan to develop a new consistency based induced ordering weighted operator in order to fuse all the expert preferences allocating more importance to those that presents higher linguistic consistency in their judgments. Moreover, the issue of the consensus[3, 5, 21] when dealing with linguistic preference relation taking into account the uncertainty in the meaning of the words will be addressed.

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